

AD-A139 259

THE DIRICHLET PROBLEM FOR HARMONIC MAPS FROM THE DISK
INTO THE EUCLIDEAN N. (U) WISCONSIN UNIV-MADISON

1/1

MATHEMATICS RESEARCH CENTER V BENCI ET AL. JAN 84

UNCLASSIFIED

MRC-TSR-2636 DAAG29-80-C-0041

F/G 12/1

NL

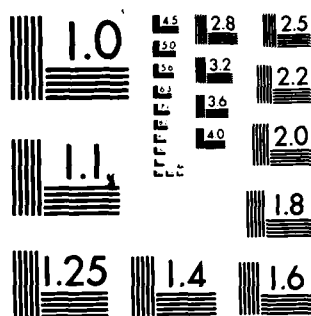
END

DATE

FILMED

4-84

DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AD A139259

MRC Technical Summary Report # 2636

THE DIRICHLET PROBLEM
FOR HARMONIC MAPS FROM THE DISK INTO
THE EUCLIDEAN n -SPHERE

V. Benci and J. M. Coron

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

January 1984

(Received October 11, 1983)

DTIC
ELECTE
MAR 22 1984
S D
A B

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

DTIC FILE COPY

84 00 21 00

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

THE DIRICHLET PROBLEM FOR HARMONIC MAPS
FROM THE DISK INTO THE EUCLIDEAN n -SPHERE

V. Benci* and J. M. Coron**

Technical Summary Report #2636

January 1984

ABSTRACT

Let $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$, $S^n = \{v \in \mathbb{R}^{n+1} \mid |v| = 1\}$ ($n \geq 2$),
and let $\gamma \in C^{2,\delta}(\partial\Omega; S^n)$. We study the following problem

$$(*) \quad \begin{cases} u \in C^2(\Omega; S^n) \cap C^0(\bar{\Omega}; S^n) \\ -\Delta u = u|\nabla u|^2 \\ u = \gamma \text{ on } \partial\Omega \end{cases}$$

Problem (*) is the "Dirichlet" problem for a harmonic function u which takes its values in S^n . We prove that, if γ is not constant, then (*) has at least two distinct solutions.

AMS (MOS) Subject Classifications: 35J65, 58E12

Key Words: Dirichlet problem, harmonic map, conformal transformation,
critical point, minimax principle, essential map.

Work Unit Number 1 - Applied Analysis

*
Università di Bari - Bari, Italy.

**
Ecole Polytechnique, Centre de Mathematiques, 91128 Palaiseau, France.

SIGNIFICANCE AND EXPLANATION

Let Ω, M be two Riemannian manifolds (Ω with boundary). A map $u : \Omega \rightarrow M$ is called harmonic if it is an extremal of the Dirichlet integral

$$(*) \quad \int_{\Omega} |\nabla u|^2 d\mu(\Omega) .$$

If $M = \mathbb{R}$, $(*)$ is the "classical" Dirichlet integral. If $\Omega = [0,1]$, the harmonic maps are the geodesics joining $u(0)$ to $u(1)$. These two situations have been studied extensively in the past. Only recently have more general situations been treated. In this paper ^{studies} we study harmonic maps when Ω is the two dimensional disk and $M = \mathbb{R}^n$. In this situation, given a smooth function γ from $\partial\Omega$ to \mathbb{R}^n we prove that if γ is not constant, there exist two harmonic functions u such that $u|_{\partial\Omega} = \gamma$.

| | |
|--------------------|-------------------------------------|
| Accession For | |
| NTIS GRA&I | <input checked="" type="checkbox"/> |
| DTIC TAB | <input type="checkbox"/> |
| Unannounced | <input type="checkbox"/> |
| Justification | |
| By | |
| Distribution/ | |
| Availability Codes | |
| Dist | Avail and/or Special |
| A-1 | |

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

THE DIRICHLET PROBLEM FOR HARMONIC MAPS
FROM THE DISK INTO THE EUCLIDEAN n -SPHERE

V. Benci* and J. M. Coron**

Introduction

Let

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

and

$$S^n = \{v \in \mathbb{R}^{n+1} \mid |v| = 1\} \quad n \geq 2.$$

Let γ be a map from $\partial\Omega$ into S^n . We seek functions u in $C^2(\Omega; S^n) \cap C^0(\bar{\Omega}; S^n)$ such that:

$$(1.1) \quad -\Delta u = u|\nabla u|^2$$

$$(1.2) \quad u = \gamma \quad \text{on } \partial\Omega.$$

We shall assume that

$$(1.3) \quad \gamma \in C^{2,\delta}(\partial\Omega) \quad \text{with } 0 < \delta < 1$$

which means that $\gamma \in C^2(\partial\Omega)$ and that the second derivative of γ is Hölder continuous with exponent δ .

The existence of at least one solution is obvious. To see this let

$$E = \{u \in H^1(\Omega; \mathbb{R}^{n+1}) \mid u|_{\partial\Omega} = \gamma, |u| = 1 \text{ a.e.}\}$$

where $H^1(\Omega; \mathbb{R}^{n+1})$ is the usual Sobolev space. Using (1.3) it is easy to see that E is non void. On E we consider the functional

$$E(u) = \int_{\Omega} |\nabla u|^2.$$

Clearly there exists some \underline{u} in E such that

$$(1.4) \quad E(\underline{u}) = \inf_E E = m.$$

\underline{u} is a solution of (1) and (2) and thanks to a result of Morrey [M2]

*
Università di Bari - Bari, Italy.

**
Ecole Polytechnique, Centre de Mathematiques, 91128 Palaiseau, France.

$$\underline{u} \in C^\infty(\Omega; S^n) \cap C^{2,\delta}(\bar{\Omega}; S^n) .$$

Our main result is:

Theorem 1

If γ is not constant then there exist at least two functions in $C^0(\bar{\Omega}; S^n) \cap C^{2,\delta}(\bar{\Omega}; S^n)$ which are solutions of (1.1) - (1.2).

Remarks

1) If $u \in C^0(\bar{\Omega}; S^n) \cap H^1(\Omega; \mathbb{R}^{n+1})$ satisfies (1.1), u is harmonic; moreover it is well known (see [LU2], [HW], [W1]) that $u \in C^\infty(\Omega; S^n)$ and if $u|_{\partial\Omega} \in C^{k,\alpha}(\partial\Omega; S^n)$, $u \in C^{k,\alpha}(\bar{\Omega}; S^n)$. In particular, in our case, if $u \in C^0(\bar{\Omega}; S^n) \cap H^1(\Omega; \mathbb{R}^{n+1})$ is a solution of (1.1) - (1.2) then $u \in C^{2,\delta}(\bar{\Omega}; S^n)$.

2) In the case $n = 2$ theorem 1 has been proved before by H. Brezis - J. M. Coron [BC2] and J. Jost [J] independently.

In this case, it is possible to assume less regularity on γ ; for example $E \neq \emptyset$ is sufficient to guarantee at least two solutions in $H^1(\Omega; S^n)$; we do not know if this is the case for $n \geq 3$. The difference between $n = 2$ and $n \geq 3$ is that E is not connected when $n = 2$ and connected when $n \geq 3$. (To see that E is connected when $n \geq 3$, use the density result due to R. Schoen - K. Uhlenbeck [SU2].)

3) When γ is constant it has been proved by L. Lemaire [LM] that, if $u \in C^0(\bar{\Omega}; S^n) \cap H^1(\Omega; \mathbb{R}^{n+1})$ is a solution of (1.1) - (1.2), then u is identically equal to the same constant.

In order to prove theorem 1 we introduce

$$(1.5) \mathcal{E}_p = \{\sigma \mid \sigma \in C^0(S^{n-2}; W_Y^{1,p}(\Omega; S^n)), \sigma \text{ is not homotopic to a constant}\}$$

where $p > 2$,

$$W_Y^{1,p}(\Omega; S^n) = \{u \mid u \in W^{1,p}(\Omega; S^n), u = \gamma \text{ on } \partial\Omega\}$$

and $C^0(S^{n-1}; W_Y^{1,p}(\Omega; S^n))$ is the set of continuous functions from S^{n-2} into $W_Y^{1,p}(\Omega; S^n)$. Let

$$(1.6) \quad \Sigma = \bigcup_{p \geq 2} \Sigma_p$$

and

$$(1.7) \quad c = \inf_{\sigma \in \Sigma} \max_{s \in S^{n-2}} E(\sigma(s)) .$$

The main result of the paper is the following theorem:

Theorem 1.2. Suppose that $\gamma \in C^{2,\delta}(\partial\Omega; S^n) (n \geq 2)$ is not constant. Then the problem (1.1), (1.2) has at least one solution $u \in C^{2,\delta}(\bar{\Omega}; S^n)$ such that $E(u) = c$; moreover if $c = m$, the problem (1.1), (1.2) has infinitely many solutions when $n \geq 3$; two solutions when $n = 2$.

Clearly theorem 1.1 follows from theorem 1.2.

The main difficulty in proving theorem 1.2 comes from a lack of compactness. For this reason we are not able to prove directly that c , defined by (1.7) is a critical value of E (i.e. that there exists u solution of (1.1), (1.2) such that $E(u) = c$). For this reason, following an idea of J. Sacks and K. Uhlenbeck [SU1] we study an approximate problem, i.e. the critical points of the functional

$$(1.8) \quad E_\alpha(u) = \int_\Omega [(1 + |Vu|^2)^\alpha - 1] dx, \quad u \in W_Y^{1,2\alpha}, \quad \alpha > 1 .$$

This functional satisfies the Palais-Smale condition. Let

$$(1.9) \quad c_\alpha = \inf_{\sigma \in \Sigma_{2\alpha}} \max_{s \in S^{n-2}} E_\alpha(\sigma(s)) .$$

We prove that c_α is a critical value of E_α larger than c and that

$$\lim_{\alpha \rightarrow 1} c_\alpha = c .$$

Just to explain the difficulty let us assume for the moment being that $c > m$.

There exists u_α such that

$$E'_\alpha(u_\alpha) = 0$$

and

$$E(u_\alpha) = c_\alpha .$$

Obviously u_α is bounded in $H^1(1)$ and therefore we can extract a subsequence u_{α_n} which converges weakly in H^1 to some u ; u satisfies (1.1) - (1.2) (see [SU1]) and the key point is to prove that $u \neq \underline{u}$. In fact we shall prove that u_{α_n} tends strongly to u and then $E(u) = c > E(\underline{u})$. The proof of the strong convergence relies on some ideas used in [BC2]. We prove the crucial strict inequality

$$c < m + 8\pi ;$$

then, using a theorem of E. Calabi [C] and arguments involved in J. Sacks - K. Uhlenbeck [SU1] we prove the strong convergence.

Remark

Similar difficulties and methods also occur in [A], [BC1], [BN], [J], [LB], [LN], [ST], [T] and [W2].

(1)

For simplicity we write H^1 instead of $H^1(\Omega; \mathbb{R}^{n+1})$.

2. A topological result

In this section we shall prove a topological result which will be used in the proof of theorem 1.2.

Let $\Omega = \{x \in \mathbb{R}^2 \mid |x| < 1\}$ and let M be a C^2 -manifold sitting in \mathbb{R}^k .

Suppose that $\gamma \in C^1(\partial\Omega; M)$ is homotopic to a constant. We set

$$H_Y^1(\Omega; M) = \{u \in H^1(\Omega; \mathbb{R}^k) \mid u|_{\partial\Omega} = \gamma \text{ and } u(x) \in M \text{ for a.e. } x \in \Omega\}$$

$$C_Y^1(\bar{\Omega}; M) = \{u \in C^1(\bar{\Omega}; M) \mid u|_{\partial\Omega} = \gamma\}.$$

For $w \in H_Y^1(\Omega; M)$ we set

$$A_\delta(w) = \{u \in H^1(\Omega; M) \mid \|u - w\|_{H^1} < \delta \text{ and } u = w \text{ on } \partial\Omega\}.$$

Theorem 2.1. For every $w \in H_Y^1(\Omega; M)$ there exist $\delta, \varepsilon_0 > 0$ and a continuous map

$$T : [0, \varepsilon_0] \times A_\delta(w) \rightarrow H_Y^1(\Omega; M)$$

such that

- (i) $T_0 u = u$ for every $u \in A_\delta(w)$
- (ii) $T_{\varepsilon_0} u \in C_Y^1(\bar{\Omega}; M)$ for every $u \in A_\delta(w)$
- (iii) $T_{\varepsilon_0} : A_\delta(w) \rightarrow C_Y^1(\bar{\Omega}; M)$ is continuous
- (iv) $T : [0, \varepsilon_0] \times [W_Y^{1,p}(\Omega; M) \cap A_\delta(w)] \rightarrow W_Y^{1,p}(\Omega; M)$ is continuous for every $p > 2$.

First we shall prove theorem 2.1 in the case in which γ is identically equal to a constant c .

Lemma 2.2. If $\gamma \equiv c$ (c is a constant) then the conclusion of theorem 2.1 holds.

Proof. We extend every map $u \in H^1_c(\Omega; M)$ to \mathbb{R}^2 taking $u(x) \equiv c$ for $x \in \mathbb{R}^2 - \Omega$. We shall denote u and its extension by the same letter.

Let $\phi \in C^\infty(\mathbb{R}^2, [0, +\infty))$ with $\int_{\mathbb{R}^2} \phi = 1$ and

$$(2.1) \quad \phi(x) = 0 \text{ if } x \notin \Omega.$$

We set

$$\phi_\varepsilon(x) = \varepsilon^{-2} \phi\left(\frac{|x|}{\varepsilon}\right)$$

and

$$(2.2) \quad u_\varepsilon(x) = (J_\varepsilon u)(x) = \int \phi_\varepsilon(x-y) u(y) dy.$$

We have the following inequality which is due to R. Schoen and K.

Uhlenbeck [SU2]: there exists $c_3 > 0$ such that $\forall \delta > 0 \exists \varepsilon_0 > 0$ such that

$$(2.3) \quad \begin{aligned} \text{dist}(u_\varepsilon(x), M) &\leq c_3 \delta \text{ for every } u \in A_\delta(w) \\ \text{for every } x \in \mathbb{R}^2, \text{ for every } \varepsilon \in [0, \varepsilon_0]. \end{aligned}$$

For the convenience of the reader we recall the proof. In fact, since $u(y) \in M$ for a.e. $y \in \mathbb{R}^2$ we have

$$\text{dist}(u_\varepsilon(x), M) \leq |u_\varepsilon(x) - u(y)|.$$

By the above formula, for $x \in \mathbb{R}^2$ we get

$$(2.4) \quad \begin{aligned} \varepsilon^2 \text{dist}(u_\varepsilon(x), M) &\leq \int_{|x-y| < \varepsilon} |u_\varepsilon(x) - u(y)| dy \leq \\ &\leq c_1 \varepsilon^2 \left[\int_{|x-y| < \varepsilon} |\nabla u(y)|^2 dy \right]^{1/2} \text{ (by the Poincaré inequality)} \\ &\leq c_1 \varepsilon^2 \left(\int_{|x-y| < \varepsilon} |\nabla u(y) - \nabla w(y)|^2 dy + \int_{|x-y| < \varepsilon} |\nabla w(y)|^2 dy \right)^{1/2} \\ &\leq c_1 \varepsilon^2 \left(\|u - w\|_{H^1(\Omega)}^2 + \int_{|x-y| < \varepsilon} |\nabla w(y)|^2 dy \right)^{1/2}. \end{aligned}$$

Since $|\nabla w|^2 \in L^1(\mathbb{R}^2)$, we can choose ε so small that

$$\int_{|x-y|<\varepsilon} |\nabla w(y)|^2 dy < \delta^2 \quad \text{for every } x \in \mathbb{R}^2.$$

So by (2.4) and the above inequalities we get

$$(2.5) \quad \text{dist}(u_\varepsilon(x), M) < c_3 \delta \quad \text{for every } u \in A_\delta(w), \text{ for } x \in \mathbb{R}^2$$

and ε sufficiently small where c_3 is a suitable constant which depends only on the Poincaré constant c_1 .

Now let d be a constant such that the projection map

$$P : N_d(M) \rightarrow M$$

is well defined. Here $N_d(M) = \{x \in \mathbb{R}^k \mid \text{dist}(x, M) < d\}$.

Now fix $\delta < \frac{d}{2c_3}$ and ε_0 small enough in order that (2.3) holds for every $\varepsilon \in (0, \varepsilon_0]$ (and every $x \in \mathbb{R}^k$, every $u \in A_\delta(w)$). Thus the map

$$P \circ J_\varepsilon : A_\delta(w) \rightarrow C^1(\mathbb{R}^2, M) \quad \varepsilon \in (0, \varepsilon_0]$$

is well defined and continuous.

Now consider the map

$$R_\varepsilon : C^1(\mathbb{R}^2, M) \rightarrow C^1(\bar{\Omega}, M)$$

defined by

$$(R_\varepsilon u)(x) = u\left(\frac{x}{1+\varepsilon}\right).$$

Clearly R_ε is continuous in u and ε . Moreover, if $u \in P \circ J_\varepsilon(A_\delta(w))$

($\varepsilon < \varepsilon_0$) it is easy to see that $(R_\varepsilon u)|_{\partial\Omega} = c$. Therefore the map

$$T : [0, \varepsilon_0] \times A_\delta(w) \rightarrow H_Y^1(\Omega; M)$$

$$T_0 = \text{Id}$$

$$T_\varepsilon = R_\varepsilon \circ P \circ J_\varepsilon$$

satisfies the requirements (i), (ii) and (iii).

Moreover one can easily check that T is continuous and moreover satisfy (iv). □

Now we shall consider the case in which γ is not constant. Since we have assumed that γ is homotopic to a constant, there exists a homotopy $h \in C^0(I \times \partial\Omega; M)$ such that

$$(2.6) \quad \begin{cases} (a) & h_0(x) = \gamma(x) \quad \forall x \in \partial\Omega \\ (b) & h_1(x) = c \quad \forall x \in \partial\Omega \quad (c \text{ is a constant}) \end{cases}$$

Since we have assumed γ to be of class C^1 , we can suppose that also h is of class C^1 .

Lemma 2.3. Under our assumptions there exist two continuous functions

$$H : I \times H_Y^1(\Omega; M) \rightarrow H^1(\Omega; M) \quad \text{with} \quad H_\lambda(u)|_{\partial\Omega} = h_\lambda(\gamma)$$

and

$$K : \{(\lambda, u) \in I \times H^1(\Omega; M) \mid u|_{\partial\Omega} = h_\lambda(\gamma)\} \rightarrow H_Y^1(\Omega; M)$$

such that

$$H_0 = K_0 = \text{identity in } H_Y^1(\Omega; M).$$

Moreover H and K are continuous also in the $W^{1,p}(\Omega; M)$ topology.

Proof. For $u \in H_Y^1(\Omega; M)$ set

$$\tilde{u}(x) = \begin{cases} u(x) & \text{for } |x| < 1 \\ h_{|x|-1}(\frac{x}{|x|}) & \text{for } 1 < |x| < 2 \end{cases}$$

By virtue of (2.6)(a) $\tilde{u} \in H_c^1(\Omega_1; M)$ where $\Omega_1 = \{x \in \mathbb{R}^2 \mid |x| < 2\}$ and of course it depends continuously on $u \in H_Y^1(\Omega; M)$.

For $v \in H_{h_\lambda(\gamma)}^1(\Omega; M)$ we set

$$\tilde{v}_\lambda = \begin{cases} v(x) & \text{for } |x| < 1 \\ h_{\lambda(2-|x|)}(\frac{x}{|x|}) & \text{for } 1 < |x| < 2 \end{cases}$$

Clearly $\tilde{v} \in H^1(\Omega; M)$.

Finally for $x \in \Omega$ set

$$(H_\lambda u)(x) = \tilde{u}((1+\lambda)x) \quad u \in H_Y^1(\Omega; M)$$

$$(K_\lambda v)(x) = \tilde{v}_\lambda((1+\lambda)x) \quad v \in H_{h_\lambda(\gamma)}^1(\Omega; M) .$$

It is easy to check that H_λ and K_λ satisfy the required conditions. \square

Proof of theorem 2.1. Let H be the map defined in lemma 2.3. Then

$$H_1(w) \in H_C^1(\Omega; M).$$

By lemma 2.2, there exists $\tilde{\delta}, \tilde{\varepsilon}_0 > 0$ and a continuous map

$$\tilde{T} : [0, \tilde{\varepsilon}_0] \times A_{\tilde{\delta}}(H_1(w)) \rightarrow H_C^1(\Omega; M)$$

which satisfies (i), (ii), (iii) and (iv) of theorem 2.1.

Since $H_1 : H_Y^1(\Omega; M) \rightarrow H_C^1(\Omega; M)$ is continuous, there exists $\delta > 0$ such that

$$H_1(A_\delta(w)) \subset A_{\tilde{\delta}}(H_1(w)) .$$

Therefore it makes sense to define a map $T : [0, 1+\varepsilon_0] \times A_\delta(w) \rightarrow H_Y^1(\Omega; M)$ as follows

$$T_\lambda(u) \begin{cases} K_\lambda \circ H_\lambda(u) & \text{for } \lambda \in [0, 1] \\ K_1 \circ \tilde{T}_{\lambda-1} \circ H_1(u) & \text{for } \lambda \in [1, 1+\tilde{\varepsilon}_0] \end{cases} .$$

Such a map satisfies (i), (ii), (iii) and (iv) of Theorem 2.1 with

$$\varepsilon_0 = 1+\tilde{\varepsilon}_0 .$$

\square

Lemma 2.3. Let $z \in C_Y^1(\bar{\Omega}; M)$ and set

$$N_\eta(z) = \{u \in C_Y^1(\bar{\Omega}; M) \mid \|z-u\|_C < \eta\} .$$

Then if η is sufficiently small, $N_\eta(z)$ is a strong deformation retract of

$\{z\}$ for every $z \in C_Y^1(\bar{\Omega}; M)$.

Proof. Choose η small enough in order that $B_\eta(y) \cap M$ is geodesically

convex in M for every $y \in M$; ($B_r(y) = \{x \in \mathbb{R}^k \mid |s-y| < r\}$). Then for

$x \in B_\eta(y)$ we define

$h_t(y, x) = \beta(t)$ where $\beta(t)$ is the (unique) geodesic on M

parametrized with the arc length such that

$$\beta(0) = y \text{ and } \beta(1) = x .$$

So if M is a smooth manifold h is smooth.

For $u \in N_\eta(z)$ we set

$$S_t(u)(x) = h_t(z(x), u(x)) .$$

Clearly $S : I \times N_\eta(z) \rightarrow C_Y^1(\bar{\Omega}; M)$ is continuous, $S_0 \equiv \text{Id}_{N_\eta(z)}$, $S_1(u) = z$ for every $u \in N_\eta(z)$ and $S_t(z) = z$ for every $t \in [0, 1]$. □

By theorem 2.1 and lemma 2.3 the following Corollary follows which will be used in the proof of our main theorem.

Corollary 2.3. For every $w \in W_Y^1(\Omega; M)$ there is $\theta > 0$ such that $A_\theta(w) \cap W^{1,p}(\Omega; M)$ is contractible to a point in $W^{1,p}(\Omega; M)$, $p > 2$.

Proof. By theorem 2.1 there exists a continuous map $T_{\varepsilon_0} : A_\delta(w) \rightarrow C_Y^1(\Omega; M)$. So given η as in lemma 2.3, there exists $\theta \in (0, \delta]$ such that

$$T_{\varepsilon_0}(A_\theta(w)) \subset N_\eta(T_{\varepsilon_0}(w)) .$$

By lemma 2.3, $N_\eta(T_{\varepsilon_0}(w))$ is contractible, then also $A_\theta(w) \cap W^{1,p}(\Omega; M)$ is contractible to a point in $W^{1,p}(\Omega; M)$. □

3. A Convergence Theorem

In order to approximate the solutions of problem (1.1), (1.2) with the critical points of the functional (1.8) we need the following theorem which has been inspired by J. Sacks and K. Uhlenbeck [SU1].

Theorem 3.1. For every $\alpha > 1$ let $u_\alpha \in E_\alpha$ be a solution of

$$(3.1) \quad E'_\alpha(u_\alpha) = 0$$

and suppose that

$$(3.2) \quad \lim_{\alpha \rightarrow 1} E(u_\alpha) < m + 8\pi.$$

Then u_α has a subsequence $u_{\alpha_k} \rightarrow u$ in $C^1(\bar{\Omega}; S^n)$ and u is a solution of (1.1).

In order to prove theorem 3.1 we need the following proposition due to J. Sacks and K. Uhlenbeck [SU1].

Proposition 3.1

There exist $\alpha_0 > 1$ such that if $u \in E_\alpha$ with $1 < \alpha \leq \alpha_0$ and $E'_\alpha(u) = 0$ then $u \in C^{2,\delta}(\bar{\Omega})$.

Proof. See the proof of proposition 2.3 in [SU1]. In fact in [SU1] only the regularity inside Ω is proved. But the theorem 1.11. 1' of Morrey [M2] which is used in [SU1] is still valid in all Ω if z is assumed to be in H_0^1 (see p. 38 in [M2]). Therefore we may apply this theorem to $z = u - \phi$ where $\phi \in C^{2,\delta}(\bar{\Omega})$ with $\phi = \gamma$ on $\partial\Omega$. We conclude that $\forall u \in H^1$. The end of the proof is an easy adaptation of the proof in [SU1].

□

Proof of theorem 3.1.

In the following we will always assume that $1 < \alpha \leq \alpha_0$. Since u_α is bounded in L^∞ and $E(u_\alpha)$ is bounded, u_α is bounded in H^1 . Therefore there exist a sequence $(\alpha_k)_{k \in \mathbb{N}}$ such that u_{α_k} tends weakly in H^1 to some u . For simplicity we shall write u_k instead of u_{α_k} . Using (3.1)

(and Proposition (3.1)) we have

$$(3.3) \quad -\Delta u_k - 2 \frac{a_k - 1}{(1 + |\nabla u_k|^2)} (\nabla u_k, \nabla u_k, \nabla^2 u_k) = u_k |\nabla u_k|^2$$

where

$$(\nabla u_k, \nabla u_k, \nabla^2 u_k) = \sum_{\substack{1 \leq i \leq 2 \\ 1 \leq j \leq 2 \\ 1 \leq p \leq n+1 \\ 1 \leq q \leq n+1}} \frac{\partial^2 u_k^p}{\partial x_i \partial x_j} \frac{\partial u_k^p}{\partial x_i} \frac{\partial u_k^q}{\partial x_j} e_q$$

$$\text{and } u_k = (u_k^1, \dots, u_k^q, \dots, u_k^{n+1}) = \sum_{q=1}^{n+1} u_k^q e_q.$$

Let

$$\theta_k = \max_{x \in \bar{\Omega}} |\nabla u_k(x)|.$$

First let us assume that θ_k is bounded.

We are going to prove that in this case u_k tends to u in $C^1(\bar{\Omega})$ and

that:

$$-\Delta u = u |\nabla u|^2.$$

Using (3.3) we have:

$$(3.4) \quad -\Delta u_k^p + (a_k - 1) \sum_{\substack{1 \leq i \leq 2 \\ 1 \leq j \leq 2 \\ 1 \leq q \leq n+1}} A_{ijk}^{pq} \frac{\partial^2 u_k^q}{\partial x_i \partial x_j} = u_k^p |\nabla u_k|^2 \quad 1 \leq p \leq n+1$$

with

$$(3.5) \quad \|A_{ijk}^{pq}\|_{C^0(\bar{\Omega})} < C.$$

Since θ_k is bounded we have:

$$(3.6) \quad \|u_k |\nabla u_k|^2\|_{C^0(\bar{\Omega})} < C.$$

It follows from (3.4), (3.5), (3.6) and a theorem of Morrey [M1] (see also [N]) that:

$$(3.7) \quad \exists \gamma > 0 \text{ such that } \|u_k\|_{C^{1,\gamma}(\bar{\Omega})} < C.$$

(Actually in [M1] and [N] the theorems are stated for one equation and not for a system. But, the proofs can be easily adapted to the system (3.4).) It follows from (3.7) that u_k tends to u in $C^1(\bar{\Omega})$. Moreover (3.3) may be written in the following divergence form:

$$-\frac{\partial}{\partial x_i} \left((1+|\nabla u_k|^2)^{\alpha_k-1} \frac{\partial u_k}{\partial x_i} \right) = u_k |\nabla u_k|^2 (1+|\nabla u_k|^2)^{\alpha_k-1}, \quad i = 1, 2.$$

Using the convergence of u_k to u in $C^1(\bar{\Omega})$ we have

$$(3.8) \quad -\Delta u = u |\nabla u|^2.$$

Now we want to show that

$$(3.9) \quad \lim_{k \rightarrow +\infty} \theta_k = +\infty.$$

is not possible. We argue indirectly and suppose that (3.9) holds. Let $a_k \in \bar{\Omega}$ such that

$$\theta_k = |\nabla u_k(a_k)|.$$

After extracting a subsequence we may assume that either

$$(3.10) \quad \lim_{k \rightarrow +\infty} \theta_k d(a_k, \partial\Omega) = +\infty$$

or

$$(3.11) \quad \lim_{k \rightarrow +\infty} \theta_k d(a_k, \partial\Omega) = \rho < +\infty$$

where $d(a_k, \partial\Omega)$ is the distance from a_k to $\partial\Omega$.

First let us assume that (3.10) holds. Then, like in [SU1] we define

$$v_k(x) = u_k\left(\frac{x}{\theta_k} + a_k\right).$$

v_k is defined on $\bar{\Omega}_k$ where

$$\Omega_k = \left\{ y \mid y \in \Omega \right\}.$$

Using (3.10) it is easy to see

$$(3.12) \quad \forall R > 0 \quad \exists k(R) \text{ such that } k > k(R) \implies B(0, R) \subset \Omega_k$$

where $B(0,R) = \{x \in \mathbb{R}^2 \mid |x| < R\}$. Moreover it follows from (3.3) that, in Ω_k ,

$$(3.13) \quad -\Delta v_k - 2 \frac{\alpha_k^{-1}}{(\theta_k^{-2} + |\nabla v_k|^2)} (\nabla v_k, \nabla v_k, \nabla^2 v_k) = v_k |\nabla v_k|^2.$$

We have

$$(3.14) \quad \|\nabla v_k\|_{C^0(\bar{\Omega}_k)} < 1.$$

As before it follows from (3.12), (3.13), (3.14) and [M1] (or [N]) that there exists $\gamma > 0$ such that $\forall R > 0$ $C(R)$ such that

$$(3.15) \quad \|v_k\|_{C^{1,\gamma}(B(0,R))} < C(R) \quad \forall k.$$

Therefore (after extracting a subsequence) we have

$$(3.16) \quad v_k \rightarrow v \quad \text{in } C^1(B(0,R)) \quad \forall R$$

and in particular

$$(3.17) \quad |\nabla v(0)| = \lim_{k \rightarrow +\infty} |\nabla v_k(0)| = 1.$$

We write (3.13) in a divergence form:

$$(3.18) \quad -\frac{\partial}{\partial x_1} \left((1 + \theta_k^2 |\nabla v_k|^2)^{\alpha_k^{-1}} \frac{\partial v_k}{\partial x_1} \right) = v_k |\nabla v_k|^2 (1 + \theta_k^2 |\nabla v_k|^2)^{\alpha_k^{-1}} \quad i = 1, 2.$$

From (3.16) and (3.18) we get

$$(3.19) \quad -\Delta v = v |\nabla v|^2.$$

Moreover

$$\int_{\Omega_k} |\nabla v_k|^2 = \int_{\Omega} |\nabla u_k|^2 < c,$$

thus

$$(3.20) \quad \int_{\mathbb{R}^2} |\nabla v|^2 < +\infty.$$

From (3.19), (3.20) and [SU1] (theorem 3.6) it follows that v can be extended to a regular harmonic map from $\mathbb{R}^2 \cup \{\infty\} = S^2$ into S^n .

The following theorem is due to E. Calabi [C] (theorem 5.5):

Theorem

Let v be a harmonic map from S^2 into S^m whose image does not lie in any equatorial hyperplane of S^m then

- i) the area $A(v)$ of $v(S^2)$ is an integer multiple of 2π
- ii) m is even, and $A(v) \geq \frac{m(m-2)}{2} \pi$.

Remark

In [C] v is assumed to be an immersion but the proof given in [C] works also if v is not an immersion (note that the points where v is not an immersion are isolated and branch points, see e.g. [GOR]).

Any harmonic map w from S^2 into S^2 which is not constant satisfy (see, for example [L] theorem (8.4))

$$E(w) \geq 8\pi.$$

Therefore if w is a harmonic map from S^2 into S^n which is not constant, using the Calabi theorem and an easy induction argument we have

$$E(w) \geq 8\pi.$$

(we recall that $E(w) \geq 2 A(w)$).

Our map v is a harmonic map from S^2 into S^n and (see (3.17)) v is not constant. Therefore

$$(3.21) \quad E(v) \geq 8\pi.$$

We are going to prove (as in [SU1]) that

$$(3.22) \quad \lim_{k \rightarrow \infty} E(u_k) \geq E(u) + E(v).$$

Since by definition of m (see (1.4))

$$(3.23) \quad E(u) \geq m,$$

using (3.21), (3.22), (3.23) and (3.2) we get a contradiction.

We may assume that a_k tends to some a in $\bar{\Omega}$. Let $\epsilon > 0$ and $r > 0$ such that

$$(3.24) \quad \int_{D(a,r)} |\nabla u|^2 < \epsilon$$

where

$$D(a,r) = \{x \in \Omega \mid |x-a| < r\}.$$

We have

$$(3.25) \quad \int_{D(a,r)} |\nabla u_k|^2 = \int_{C_k} |\nabla v_k|^2$$

where

$$C_k = \{\theta_k^{1/2} (y-a_k) \mid y \in D(a,r)\}.$$

Using (3.10) we have

$$\forall R > 0 \exists k(R) \text{ such that } k > k(R) \implies B(0,R) \subset C_k.$$

Therefore

$$(3.26) \quad \lim_{k \rightarrow +\infty} \int_{C_k} |\nabla v_k|^2 > E(v).$$

From (3.24), (3.25) and (3.26) we have

$$\lim_{k \rightarrow +\infty} E(u_k) > E(u) + E(v) - \epsilon \quad (\forall \epsilon > 0)$$

which proves (3.22).

Now it remains to exclude (3.11). We assume (3.11) holds. Now (3.12) is false.

We may assume that a_k tends to some a . Using (3.5) and (3.9) we see that $a \in \partial\Omega$; with no lack of generality we may assume that

$$\lim_{k \rightarrow +\infty} a_k = (-1, 0) = a.$$

Let $T : \mathbb{R}^2 - \{(1,0)\} \rightarrow \mathbb{R}^2$

$$(3.27) \quad T(x_1, x_2) = \left(-\frac{x_1-1}{(x_1-1)^2 + x_2^2}, \frac{x_2}{(x_1-1)^2 + x_2^2} \right) = (\bar{x}_1, \bar{x}_2).$$

T is a conformal diffeomorphism between $\Omega - \{(1,0)\}$ and $] \frac{1}{2}, +\infty[\times \mathbb{R}$ and

$$T(\partial\Omega - \{(1,0)\}) = \{\frac{1}{2}\} \times \mathbb{R}$$

$$(3.28) \quad T^{-1}((\bar{x}_1, \bar{x}_2)) = (1 - \frac{\bar{x}_1}{\bar{x}_1^2 + \bar{x}_2^2}, \frac{\bar{x}_2}{\bar{x}_1^2 + \bar{x}_2^2})$$

Let $U =]\frac{1}{2}, +\infty[\times \mathbb{R}$ and let

$$\bar{u}_k = u_k \circ T^{-1}$$

Clearly

$$\bar{u}_k \in C^1(\bar{U})$$

and a straightforward computation yields

$$\Delta u_k(x) = |\bar{x}|^4 \Delta \bar{u}_k(\bar{x})$$

$$|\nabla u_k|^2(x) = |\bar{x}|^4 |\nabla \bar{u}_k|^2(\bar{x})$$

where $\bar{x} = Tx$.

In particular:

$$(3.29) \quad \|\nabla \bar{u}_k\|_{C^0(\bar{U})} \leq 4 \theta_k$$

and

$$(3.30) \quad |\nabla \bar{u}_k(\bar{a}_k)| = \frac{\theta_k}{|\bar{a}_k|^2} \sim 4\theta_k \text{ as } k \rightarrow \infty,$$

where $\bar{a}_k = Ta_k$.

Using (3.3) we get ($1 \leq p \leq n$):

$$(3.31) \quad -\Delta \bar{u}_k^p + (\alpha_k - 1) \sum_{\substack{1 \leq i \leq 2 \\ 1 \leq j \leq 2 \\ 1 \leq q \leq n+1}} B_{ijk}^{pq} \frac{\partial^2 \bar{u}_k^q}{\partial x_i \partial x_j} =$$

$$\bar{u}_k^p |\nabla \bar{u}_k|^2 + (\alpha_k - 1) \sum_{\substack{1 \leq i \leq 2 \\ 1 \leq q \leq n+1}} C_{ik}^{pq} \frac{\partial \bar{u}_k^q}{\partial x_i}$$

where

$$B_{ijk}^{pq} \in C^0(\bar{U}), C_{ik}^{pq} \in C^0(\bar{U})$$

and

$$(3.32) \quad \|B_{ijk}^{pq}\|_{C^0(\bar{U})} < C, \|C_{ik}^{pq}\|_{C^0(\bar{U})} < C.$$

We have

$$\bar{u}_k = \bar{\gamma} \text{ on } \partial\Omega \text{ with}$$

$$\bar{\gamma}(\frac{1}{2}, t) = \gamma(\frac{4t^2-1}{4t^2+1}, \frac{4t}{4t^2+1}).$$

If $\bar{a}_k = (\bar{x}_k, \bar{y}_k)$ and $a_k = (x_k, y_k)$, using (3.28) we get:

$$\begin{aligned} \bar{x}_k - \frac{1}{2} &= \frac{1-x_k}{(x_k-1)^2 + y_k^2} - \frac{1}{2} \\ &= \frac{1-(x_k^2+y_k^2)}{2[(x_k-1)^2 + y_k^2]}. \end{aligned}$$

Then, by (3.11), we have:

$$(3.33) \quad \bar{x}_k = \frac{1}{2} + \frac{\rho}{4\theta_k} + \frac{o(1)}{\theta_k} \quad (k \rightarrow +\infty).$$

Let

$$\hat{u}_k(\hat{x}, \hat{y}) = \bar{u}_k(\frac{1}{2} + \frac{1}{\theta_k}(\hat{x} - \frac{1}{2}), \frac{\hat{y}}{\theta_k} + y_k).$$

We have $\hat{u}_k = \hat{\gamma}_k$ on $\partial\Omega$ with

$$\hat{\gamma}_k(\frac{1}{2}, t) = \bar{\gamma}(\frac{1}{2}, \frac{t}{\theta_k} + y_k)$$

and thus

$$(3.34) \quad \hat{\gamma}_k \rightarrow \bar{\gamma}(\frac{1}{2}, 0) \text{ in } C^{2,\delta}(\partial U).$$

Using (3.29) we have

$$(3.35) \quad \|\nabla \hat{u}_k\|_{C^0(\bar{U})} < 4.$$

Using (3.31) and (3.32) we have (for $1 < p < n$):

$$-\Delta \tilde{u}_k^p + (\alpha_k - 1) \sum_{\substack{1 \leq i \leq 2 \\ 1 \leq j \leq 2 \\ 1 \leq q \leq n+1}} \tilde{B}_{ijk}^{pq} \frac{\partial^2 \tilde{u}_k^q}{\partial x_i \partial x_j} =$$

(3.36)

$$\tilde{u}_k^p |\nabla \tilde{u}_k|^2 + \frac{1}{\theta_k} (\alpha_k - 1) \sum_{\substack{1 \leq i \leq 2 \\ 1 \leq q \leq n+1}} \tilde{C}_{ik}^{pq} \frac{\partial \tilde{u}_k^q}{\partial x_i}$$

with

$$(3.37) \quad \|\tilde{B}_{ijk}^{pq}\|_{C^0(\bar{U})} < C, \quad \|\tilde{C}_{ik}^{pq}\|_{C^0(\bar{U})} < C.$$

Let $R > 0$ and $U_R = U \cap \{x \in \mathbb{R}^2 \mid |x| < R\}$. Using (3.34), (3.35), (3.36), (3.37) and the Morrey-Nirenberg estimate [M1], [N] we get:

$$(3.38) \quad \exists \alpha > 0 \quad \exists C(R) \text{ such that } \|\tilde{u}_k\|_{C^{1,\alpha}(\bar{U}_R)} < C(R), \quad \forall k.$$

Remark

Actually in [M1] there is not estimate up to the boundary but this estimate can be deduced from the interior estimate, see [GT] (p. 248-249). One can find estimate up to the boundary in [LU1] (p. 455-456) and [N]. In all these references the theorems are stated for only one equation but the proofs can be easily adapted to our system (3.36).

Therefore we may assume that for some \tilde{u} in $C^{1,\alpha}(\bar{U})$:

$$(3.39) \quad \lim_{k \rightarrow +\infty} \|\tilde{u}_k - \tilde{u}\|_{C^1(\bar{U}_R)} = 0.$$

Moreover, using (3.36), (3.37), (3.35) it is easy to see that if ω is a bounded regular open set of U such that $\bar{\omega} \subset U$ then

$$\|\tilde{u}_k\|_{W^{2,2}(\omega)} < C(\omega).$$

Therefore, using (3.36) we have:

$$(3.40) \quad -\Delta \tilde{u} \approx \tilde{u} |\nabla \tilde{u}|^2 \quad \text{in } U.$$

With (3.34) we get

$$(3.41) \quad \tilde{u} = \bar{\gamma}(\frac{1}{2}, 0) \quad \text{on } \partial U.$$

Moreover

$$\int_{\Omega} |\nabla u_k|^2 = \int_U |\nabla \bar{u}_k|^2 = \int_U |\nabla \tilde{u}_k|^2,$$

therefore:

$$(3.42) \quad \int_U |\nabla \tilde{u}|^2 < +\infty.$$

We recall that $\tilde{u} \in C^0(\bar{U})$ (and even $\in C^{1,\alpha}(\bar{U})$). Then using (3.40), (3.41), (3.42) and a very slight modification of a theorem of L. Lemaire (see the appendix) we have

$$(3.43) \quad \tilde{u} \equiv \bar{\gamma}(\frac{1}{2}, 0).$$

But, using (3.30):

$$(3.44) \quad \lim_{k \rightarrow +\infty} |\nabla \tilde{u}_k(\theta_k(\bar{x}_k - \frac{1}{2}) + \frac{1}{2}, 0)| = \sqrt{2}$$

and using (3.33):

$$(3.45) \quad \lim_{k \rightarrow +\infty} \theta_k(\bar{x}_k - \frac{1}{2}) + \frac{1}{2} = \frac{1}{2} + \frac{\rho}{4}$$

and then using (3.39), (3.43), (3.44), (3.45) we get a contradiction.

□

4. Proof of Theorem 1.2

The proof of theorem 1.2 lies on several lemmas.

Lemma 4.1. Let m and c be the constants defined by (1.4) and (1.7) respectively. Then

$$c < m + 8\pi.$$

Proof. We shall construct a map $\sigma_\varepsilon \in \Sigma_3$ such that

$$(4.1) \quad E(\sigma_\varepsilon(s)) < m + 8\pi.$$

Then the conclusion follows by the definition of c . The construction of such a map is an adaptation of the proof of lemma 2 in [BC2].

Let $\underline{u} \in E$ such that $E(\underline{u}) = m$. Thanks to Morrey's regularity result $\underline{u} \in C^\infty(\bar{\Omega}; \mathbb{R}^{n+1}) \cap C^{2,\varepsilon}(\bar{\Omega}; \mathbb{R}^{n+1})$. Since γ is not constant \underline{u} is not constant and therefore $\nabla \underline{u}(x_0, y_0) \neq 0$ for some (x_0, y_0) in $\bar{\Omega}$; rotating coordinates in \mathbb{R}^2 we may always assume that

$$\underline{u}_x(x_0, y_0) \cdot \underline{u}_y(x_0, y_0) = 0.$$

Let $(e_i)_{1 \leq i \leq n+1}$ be an orthonormal basis in \mathbb{R}^{n+1} such that:

$$\underline{u}_x(x_0, y_0) = ae_1$$

$$\underline{u}_y(x_0, y_0) = be_2$$

$$\underline{u}(x_0, y_0) = e_3$$

with $a > 0$, $b > 0$, $a+b > 0$.

We shall identify S^{n-2} to $S^n \cap \{v \in S^n \mid v \cdot e_1 = 0, v \cdot e_2 = 0\}$. Let r and θ be such that $x - x_0 = r \cos \theta$, $y - y_0 = r \sin \theta$. Let $\varepsilon > 0$ be small enough. Let $\lambda = \frac{1}{2} \varepsilon^2 \max(a, b) > 0$.

We define a map $\sigma_\varepsilon \in C^0(S^{n-2}; W_Y^{1,3}(\bar{\Omega}; S^n))$ in the following way (where $s \in S^{n-2}$):

if $2\varepsilon < r$, $\sigma_\varepsilon(s)(x, y) = \underline{u}(x, y)$

$$\text{if } \lambda < r < \varepsilon, \sigma_\varepsilon(s)(x, y) = \frac{2\lambda}{\lambda^2 + r^2} (x - x_0)e_1 + \frac{2\lambda}{\lambda^2 + r^2} (y - y_0)e_2 + \frac{r^2 - \lambda^2}{\lambda^2 + r^2} e_3$$

$$\text{if } r < \lambda, \sigma_\epsilon(s)(x,y) = \frac{2\lambda}{\lambda^2+r^2} (x-x_0)e_1 + \frac{2\lambda}{\lambda^2+r^2} (y-y_0)e_2 + \frac{r^2-\lambda^2}{\lambda^2+r^2} s$$

$$\text{if } \epsilon < r < 2\epsilon, \sigma_\epsilon(s)(x,y) = \sum_{i=1}^{n+1} (A_i r + B_i) e_i + [1 - \sum_{i=1}^{n+1} (A_i r + B_i)^2]^{1/2} e_3$$

where A_i and B_i depend only on θ and ϵ and are such that $\sigma_\epsilon(s)$ is continuous at $r = \epsilon$ and $r = 2\epsilon$ for each s . More precisely

$$2\epsilon A_i + B_i = \underline{u}^i(x_0 + 2\epsilon \cos \theta, y_0 + 2\epsilon \sin \theta), \quad 1 \leq i \leq n+1$$

$$\epsilon A_1 + B_1 = \frac{2\lambda\epsilon}{\lambda^2+\epsilon^2} \cos \theta$$

$$\epsilon A_2 + B_2 = \frac{2\lambda\epsilon}{\lambda^2+\epsilon^2} \sin \theta$$

$$\epsilon A_i + B_i = 0, \quad 3 \leq i \leq n+1.$$

Since $\underline{u} \in W^{1,3}(\Omega; S^n)$, $\sigma_\epsilon \in C^0(S^{n-2}, W_Y^{1,3}(\Omega; S^n))$. Moreover

$$E(\sigma_\epsilon(s)) = E(\sigma_\epsilon(e_3)) \quad \text{for every } s \in S^{n-2},$$

and a straightforward computation leads to

$$E(\sigma_\epsilon(e_3)) = E(\underline{u}) + 8\pi - v\epsilon^2 + o(\epsilon^2), \quad (\epsilon \rightarrow 0),$$

where $v > 0$ (see [BC2]).

Therefore we can fix ϵ small enough in order that

$$E(\sigma(s)) < E(\underline{u}) + 8\pi$$

where $\sigma = \sigma_\epsilon$.

It remains to prove that $\sigma \in \Sigma_\alpha$ ($1 < \alpha < \frac{3}{2}$) i.e. that σ is an essential map. We argue indirectly. Suppose that σ is not essential. Then there exists a continuous map $\bar{\sigma}$

$$\bar{\sigma} : I \times S^{n-2} \rightarrow W_Y^{1,2\alpha}(\Omega, S^n) \quad (I = [0,1])$$

such that

$$\bar{\sigma}(0, \cdot) = \sigma(\cdot);$$

$$\bar{\sigma}(1, s) = \underline{u}$$

for every $s \in S^{n-2}$ where $\underline{u} \in W_Y^{1,2\alpha}(\Omega, S^n)$.

Now we define $\eta : I \times \bar{\Omega} \times S^{n-2} \rightarrow S^n$ as follows:

$$\eta(t, x, y, s) = \bar{\sigma}(t, s)(x, y) .$$

Clearly η is continuous in all its variables and we have:

$$(4.2) \quad \begin{cases} (a) & \eta(0, x, y, s) = \sigma(s)(x, y) \\ (b) & \eta(1, x, y, s) = u(x, y) \\ (c) & \eta(t, x, y, s) = \gamma(x, y) \quad \forall (x, y) \in \partial\Omega, \forall t \in I, \forall s \in S^{n-2} . \end{cases}$$

Next step will be to extend η to a map

$$\zeta : I \times \partial(\Omega \times B^{n-1}) \rightarrow S^n$$

as follows

$$\zeta(t, x, y, s) = \begin{cases} \eta(t, x, y, s) & \text{if } (x, y) \in \Omega \text{ and } s \in \partial B^{n-1} = S^{n-2} \\ \gamma(x, y) & \text{if } (x, y) \in \partial\Omega \text{ and } s \in B^{n-1} . \end{cases}$$

By (4.2)(c) it follows that ζ is continuous. Since $\partial(\Omega \times B^{n-1})$ is topologically equivalent to S^n the topological degree of $\zeta(t, \cdot)$ is well defined for every $t \in I$. We shall compute it for $t = 0$ and $t = 1$. To this end we extend $\zeta(t, \cdot)$ to a map

$$\theta(t, \cdot) : \bar{\Omega} \times B^{n-1} \rightarrow R^{n+1}$$

since

$$(4.3) \quad \deg(\zeta(t, \cdot)) = \deg(\theta(t, \cdot), \Omega \times B^{n-1}, w)$$

for every $w \in \text{int}(B^{n+1})$. For $t = 1$ we set

$$\theta(1, x, y, z) = u(x, y) .$$

Then by (4.3) it follows that

$$(4.4) \quad \deg(\zeta(1, \cdot)) = 0$$

since $\theta(1, x, y, z)$ is independent of z . For $t = 0$ we set

$$\theta(0, x, y, z) = \begin{cases} \eta(0, x, y, s_0) = \sigma(s_0)(x, y) & \text{if } r > \lambda, s_0 \in S^{n-2} \text{ fixed} \\ \frac{2\lambda}{\lambda^2 + r^2} (x - x_0)e_1 + \frac{2\lambda}{\lambda^2 + r^2} e_2 + \frac{r^2 - \lambda^2}{\lambda^2 + r^2} z & \text{if } r < \lambda \end{cases}$$

where $r = [(x - x_0)^2 + (y - y_0)^2]^{1/2}$ and we shall compute

$$\deg(\theta(0, \cdot), \Omega \times B^{n-1}, w) \text{ with } w = x_0 e_1 + y_0 e_2.$$

First notice that $|w| = [|x_0|^2 + |y_0|^2]^{1/2} < 1$, so the degree is well defined and it is equal to the algebraic sum of the nondegenerate solutions of the equation

$$(4.5) \quad \begin{cases} (x, y, z) \in \bar{\Omega} \times B^{n-1} \\ \theta(0, x, y, z) = w \end{cases}$$

Since $|w| < 1$ and $|\theta(0, x, y, z)| = 1$ for $|(x, y)| > \lambda$ the solutions of (4.5) are the same that the solutions of the following equation

$$(4.6) \quad \begin{cases} (x, y, z) \in \bar{\Omega} \times B^{n-1} \\ |(x, y)| < \lambda \\ \frac{2\lambda}{\lambda^2 + r^2} [(x - x_0)e_1 + (y - y_0)e_2] + \frac{r^2 - \lambda^2}{\lambda^2 + r^2} z = w \end{cases}$$

By inspection we see that the only solution of (4.6) is $x = y = z = 0$, and that it is not degenerate. Therefore $\deg(\zeta(0, \cdot)) = \pm 1$ and this contradicts (4.4). □

We now set

$$(4.7) \quad c_\alpha = \inf_{\sigma \in \Sigma_{2\alpha}} \sup_{s \in S^{n-2}} E_\alpha \circ \sigma(s)$$

where $\Sigma_{2\alpha}$ is defined by (1.5).

Lemma 4.2. For every $\alpha > 1$, the c_α 's defined by (4.7) are critical values of E_α . Moreover $c_\alpha \rightarrow c$ for $\alpha \rightarrow 1$ and $c_\alpha > c$.

Proof. It is straightforward to check that E_α satisfies the assumption (c) of Palais-Smale on E_α . Then by well known facts about the critical point theory the c_α 's are critical values of E_α .

Now we shall prove the second statement. Since $E_\alpha(u) > E(u)$ for every $u \in E_\alpha$, we have that

$$\begin{aligned} c_\alpha &> \inf_{\sigma \in \Sigma_{2\alpha}} \sup_{s \in S^{n-2}} E \circ \sigma(s) \\ &> \inf_{\sigma \in \Sigma} \sup_{s \in S^{n-2}} E \circ \sigma(s) = c \quad (\text{since } \Sigma_{2\alpha} \subset \Sigma) . \end{aligned}$$

Thus $c_\alpha > c$ for every $\alpha > 1$.

Now let us prove that $c_\alpha \rightarrow c$. Choose $\varepsilon > 0$. Then there exists $p > 2$ and $\bar{\sigma} \in \Sigma_p$ such that

$$(4.8) \quad c + \varepsilon > \sup_{u \in \bar{\sigma}(S^{n-2})} E(u) .$$

For $u \in \bar{\sigma}(S^{n-2}) \subset E_\alpha$ with $\alpha < p/2$ we have

$$\frac{d}{d\alpha} E_\alpha(u) = \int_{\Omega} (1 + |\nabla u|^2)^\alpha \log(1 + |\nabla u|^2) dx .$$

In particular, if we fix $\alpha_0 < p/2$ we have that the function

$(\alpha, s) \mapsto \frac{d}{d\alpha} E_\alpha(\bar{\sigma}(s))$ is bounded by a constant M in $[1, \alpha_0] \times S^{n-2}$. Thus, for $u \in \bar{\sigma}(S^{n-2})$ we have

$$E_\alpha(u) \leq E(u) + (\alpha-1) \left| \frac{d}{d\alpha} E_\alpha(u) \right| \leq E(u) + (\alpha-1)M .$$

We now choose $\bar{\alpha}$ such that $E_\alpha(u) \leq E(u) + \varepsilon \quad \forall u \in \bar{\sigma}(S^{n-2}) \quad \forall \alpha < \bar{\alpha}$. Then by

(4.8), $\forall \alpha < \bar{\alpha}$

$$c + \varepsilon > \sup_{s \in S^{n-2}} (E_\alpha \circ \bar{\sigma}(s) - \varepsilon) = \sup_{s \in S^{n-2}} E_\alpha \circ \bar{\sigma}(s) - \varepsilon >$$

$$> \inf_{\sigma \in \Gamma_{2\alpha}^-} \sup_{s \in S^{n-2}} E_\alpha \circ \sigma(s) - \varepsilon = c_\alpha - \varepsilon, \text{ i.e. } c_\alpha < c + 2\varepsilon \quad \forall \alpha < \bar{\alpha}.$$

□

Finally we can prove the theorem 1.2.

Proof of theorem 1.2. We consider two cases: $c > m$ and $c = m$.

I case $c > m$. For $\alpha > 1$, let u_α be a solution of $E'_\alpha(u_\alpha) = 0$ which exists by lemma 4.7. Also by lemma 4.2 and 4.1, it follows that

$$\lim_{\alpha \rightarrow 1} E_\alpha(u_\alpha) = \lim_{\alpha \rightarrow 1} c_\alpha = c < m + 8\pi$$

and since $m < E(u_\alpha) < E_\alpha(u_\alpha)$ we have that

$$\lim_{\alpha \rightarrow 1} E(u_\alpha) = c < m + 8\pi.$$

Then the conclusion follows from theorem 3.1.

□

II case $c = m$. Choose $\varepsilon > 0$, then there exists $\sigma_\varepsilon \in \Gamma$ such that

$$(4.9) \quad \max_{s \in S^{n-2}} E \circ \sigma(s) < m + \varepsilon.$$

Let $\underline{u}_\varepsilon$ be such that $E(\underline{u}_\varepsilon) = \min_{s \in S^{n-2}} E \circ \sigma(s).$

We consider a subsequence $\underline{u}_{\varepsilon_k} (\varepsilon_k \rightarrow 0)$ (which for simplicity will be denoted \underline{u}_k) which converges weakly to some \underline{u} . Since $\lim_{k \rightarrow +\infty} E(\underline{u}_k) = m$, and since E is weakly lower semicontinuous it follows that

$$\lim_{k \rightarrow +\infty} E(\underline{u}_k) = E(\underline{u}).$$

The above equality and the weak convergence $\underline{u}_k \rightharpoonup \underline{u}$ imply that $\underline{u}_k \rightarrow \underline{u}$ strongly in H^1 . By Corollary 2.3 we can choose $\delta_0 > 0$ such that $A_{\delta_0, \gamma}(\underline{u})$ is contractible in $W_Y^{1, 2\alpha}(\Omega; M).$

We claim that for every $\delta < \delta_0$ and ε_k small enough there is $\delta_k \in \sigma_{\varepsilon_k}(S^{n-2})$ such that

$$(4.10) \quad \|\underline{u}_k - u_k^\delta\|_{H^1} = \delta.$$

In fact, if the above equality does not hold, then

$$\sigma_{\varepsilon_k}(S^{n-2}) \subset \Lambda_{\delta_0, \gamma}(\underline{u})$$

and this is absurd since σ_{ε_k} is an essential map. Therefore, by (4.9) with

$\varepsilon = \varepsilon_k$, we get

$$\lim_{k \rightarrow +\infty} E(u_k^\delta) = m$$

and since E is weakly lower semicontinuous we get that

$$(4.11) \quad \lim_{k \rightarrow +\infty} E(u_k^\delta) = E(u^\delta)$$

where u^δ is the weak limit of u_k^δ (may be after having taken a subsequence). By the weak convergence of u_k^δ and (4.11), it follows that $u_k^\delta \rightarrow u^\delta$ strongly in H^1 . So taking the limit in (4.10) we get

$$\|\underline{u} - u^\delta\| = \delta.$$

Thus, for any $\delta \in [0, \delta_0)$ we get at least one solution \underline{u}^δ of our problem. \square

APPENDIX

Let $\omega = (0, +\infty) \times \mathbb{R}$ and $u \in C^0(\bar{\omega}; S^n)$ such that

$$(A.1) \quad \nabla u \in L^2(\omega)$$

$$(A.2) \quad -\Delta u = u|\nabla u|^2$$

$$(A.3) \quad \exists P \in S^n \text{ such that } u = P \text{ on } \partial\omega,$$

then

$$(A.4) \quad u \equiv P \text{ in } \omega.$$

Remarks

1. When ω is a bounded contractible open set of \mathbb{R}^2 (A.4) is also true; this theorem is due to L. Lemaire [LM] (Theoreme (3.2)). However, we cannot obtain (A.4) from the result of L. Lemaire and a conformal change of the variable. In fact consider a diffeomorphism I between ω and Ω (the open unit disk of \mathbb{R}^2) such that (for example)

$$I(\partial\omega) = \partial\Omega - \{(0, 1)\}.$$

Let

$$v = u \circ I^{-1}.$$

Clearly we have:

$$v \in C^0(\bar{\Omega} - \{(0, 1)\}; S^n)$$

$$\int_{\Omega} |\nabla v|^2 < +\infty$$

$$-\Delta v = v|\nabla v|^2$$

$$v = P \text{ on } \partial\Omega.$$

But we cannot apply directly the theorem of Lemaire since we do not know if $v \in C^0(\bar{\Omega}; S^n)$.

2. Thanks to a classical theorem (see, for example [HH], [LU2] pp. 485-493) using (A.1), (A.2), (A.3) and $u \in C^0(\bar{\omega}, S^n)$ we know that u is analytic in $\bar{\Omega}$.

3. Our proof of (A.4) is inspired from H. Wente [W1].

Proof of (A.4)

We may assume that $P = e_{n+1}$. Let w be the following function from \mathbb{R}^2 into S^n :

$$\begin{aligned} \text{if } x > 0 & \quad w(x, y) = u(x, y) \\ \text{if } x < 0 & \quad w^p(x, y) = -u^p(-x, y) \text{ for } 1 \leq p \leq n \\ & \text{and} \\ & \quad w^{n+1}(x, y) = u^{n+1}(-x, y) . \end{aligned}$$

Since $|u|^2 = 1$ and $u(0, y) = P \quad \forall y \in \mathbb{R}$ we have:

$$(A.5) \quad \frac{\partial}{\partial x} w^{n+1}(0, y) = 0 \quad \forall y \in \mathbb{R} .$$

Then, using (A.2), (A.3) and (A.5), it is easy to see that

$$(A.6) \quad -\Delta w = w|\nabla w|^2 \quad (\text{in the distribution sense}) .$$

Moreover $w \in C^0(\mathbb{R}^2) \cap H_{loc}^1(\mathbb{R}^2)$. Thus (see [LU2], [W1] or [HW]) w is analytic.

Let $\phi \in C^\infty(\mathbb{R}^2, \mathbb{C})$ be defined by:

$$\phi = w_x^2 - w_y^2 - 2i w_x \cdot w_y .$$

Using (A.6) and $|w| = 1$ it is easy to see that ϕ is holomorphic.

Moreover, by (A.1), we have $\phi \in L^1(\mathbb{R}^2)$ and, therefore $\phi = 0$. Hence

$$\nabla w = 0 \quad \text{on } \{0\} \times \mathbb{R} ,$$

which implies

$$w \equiv P \quad \text{in } \mathbb{R}^2 .$$

REFERENCES

- [A] Th. Aubin, Equations différentielles non linéaires et problème de Yamabe concernant la curbure scalaire, J. Math. Pures et Appl. 55 (1976), 269-296.
- [BC1] H. Brézis - J. M. Coron, Multiple solutions of H-systems and Rellich's conjecture, Comm. Pure Appl. Math. (to appear).
- [BC2] H. Brézis - J. M. Coron, Large solutions for harmonic maps in two dimensions, Comm. Math. Phys. 92 (1983), 203-215.
- [BN] H. Brézis L. Nirenberg, Postive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. XXXVI (1983), 437-477.
- [C] E. Calabi, Minimal immersions of surfaces in Eudlidean spheres, J. Diff. Geometry, 1 (1967), 111-125.
- [GOR] R. D. Gulliver - R. Osserman - H. L. Royden, A theory of branched immersions of surfaces, Amer. J. Math. 95 (1973), 750-780.
- [GT] D. Gilbarg - N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin-Heidelberg-New York (1977).
- [HW] S. Hildebrandt - K. O. Widman, Some regularity results for quasilinear elliptic systems of second order, Math. Z. 142 (1975), 67-86.
- [J] J. Jost, The Dirichlet problem for harmonic maps from a surface with boundary onto a 2-sphere with nonconstant boundary values, J. Diff. Geometry (to appear).
- [LU1] O. A. Ladyzenskaya - N. N. Ural'ceva, Linear and Quasilinear Elliptic Equations, New York and London: Academic Press (1968).
- [LU2] O. A. Ladyzenskaya - N. N. Ural'ceva, Linear and quasilinear elliptic equations, second Russian edition, Moscow: Nauka (1973).

- [LM] L. Lemaire, Applications harmoniques de surfaces riemanniannes, J. Diff. Geometry 13 (1978), 51-78.
- [LB] E. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, Annals of Math. (to appear).
- [LN] P. L. Lions, The concentration compactness principle in the calculus of variations; the limit case (to appear).
- [M1] C. B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc. 43 (1938), 126-166.
- [M2] C. B. Morrey, Multiple Integrals in the Calculus of Variations, Springer-Verlag, Berlin-Heidelberg-New York (1966).
- [N] L. Nirenberg, On nonlinear elliptic partial differential equations and Hölder continuity, Comm. Pure App. Math. 6 (1953), 103-156.
- [SU1] J. Sacks - K. Uhlenbeck, The existence of minimal immersions of 2-spheres, Annals of Math. 113 (1981), 1-24.
- [SU2] R. Schoen - K. Uhlenbeck, Boundary regularity and the Dirichlet problem harmonic maps, J. Diff. Geometry 18 (1983), 253-268.
- [ST] M. Struwe, Nonuniqueness in the Plateau problem for surfaces of constant mean curvature (to appear).
- [T] C. Taubes, The existence of a non-minimal solution to the SU(2) Yang-Mills-Higgs equations on R^3 (to appear).
- [W1] H. Wente, The differential equation $\Delta x = 2Hx_u \wedge x_v$ with vanishing boundary values, Proc. A.M.S., 50 (1975), 131-137.
- [W2] H. Wente, The Dirichlet problem with a volume constraint, Manuscripta Math. 11 (1974), 141-157.
- [W1] M. Wiegner, A-priori Schranken für Lösungen gewisser elliptischer Systeme, Math. Z. 147 (1976), 21-28.

VB/JMC/jvs

| REPORT DOCUMENTATION PAGE | | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------|------------------------------------------------------------------------------------------------------|
| 1. REPORT NUMBER #2636 | 2. GOVT ACCESSION NO. AD-A139259 | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle) The Dirichlet Problem for Harmonic Maps from the Disk into the Euclidean n-Sphere | | 5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period |
| 7. AUTHOR(s) V. Benci and J. M. Coron | | 6. PERFORMING ORG. REPORT NUMBER |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706 | | 8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041 |
| 11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P. O. Box 12211 Research Triangle Park, North Carolina 27709 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) | | 12. REPORT DATE January 1984 |
| | | 13. NUMBER OF PAGES 31 |
| | | 15. SECURITY CLASS. (of this report) UNCLASSIFIED |
| 16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited. | | 18a. DECLASSIFICATION/DOWNGRADING SCHEDULE |
| 17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report) | | |
| 18. SUPPLEMENTARY NOTES | | |
| 19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Dirichlet problem, harmonic map, conformal transformation, critical point, minimax principle, essential map | | |
| 20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Let $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$, $S^n = \{v \in \mathbb{R}^{n+1} \mid v = 1\}$ ($n \geq 2$), and let $\gamma \in C^{2,\delta}(\partial\Omega; S^n)$. We study the following problem $(*) \quad \begin{cases} u \in C^2(\Omega; S^n) \cap C^0(\bar{\Omega}; S^n) \\ -\Delta u = u Vu ^2 \\ u = \gamma \text{ on } \partial\Omega \end{cases}$ | | |

ABSTRACT (continued)

Problem (*) is the "Dirichlet" problem for a harmonic function u which takes its values in S^n . We prove that, if γ is not constant, then (*) has at least two distinct solutions.